International Journal of Theoretical Physics, Vol. 22, No. 3, 1983

Conformally Flat Null Electromagnetic Field

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Received April 6, 1981

A class of conformally flat solutions for null electromagnetic field is presented. Explicit forms of the field tensors are also given. The null field is characterized by shear-free, twist-free, expansion-free, and geodetic null congruences.

1. INTRODUCTION

While there do not exist free gravitational fields which are conformally flat (Kasner, 1921), one can have conformally flat solutions of Einstein's field equations with nonvanishing energy-momentum tensor. There are some systematic investigations of such solutions in the literature (Fock, 1965; Vaschek, 1969). We have obtained in this paper a class of conformally flat solutions corresponding to null electromagnetic field. The metric in this case is taken to be $g_{\mu\nu} = e^{\sigma}\eta_{\mu\nu}$ with $\eta_{\mu\nu}$ representing the Minkowskian metric. It is evident (Eisenhart, 1949) that here we have the vanishing Weyl tensor, that is

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \tfrac{1}{2}\left(g_{\mu\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu}\right) + \tfrac{1}{6}\left(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu}\right)R = 0$$

The metric in our case is found to be a linear function of the coordinates and is finally interpreted as due to a plane wave with the propagation vector constituting a geodetic, shear-free, twist-free, and hypersurface orthogonal null congruence. We have shown that such solutions do indeed correspond to null electromagnetic fields satisfying Einstein–Maxwell equations. The explicit forms of the electromagnetic field tensors $F^{\mu\nu}$ are also given.

2. FIELD EQUATIONS AND THEIR SOLUTIONS

We consider the metric in the form

$$g_{\mu\nu} = e^{\sigma}\eta_{\mu\nu} \tag{1}$$

where $\eta_{\mu\nu}$ represents the Minkowski metric. The Ricci tensor $R_{\mu\nu}$ is then given by (Eisenhart, 19xx)

$$R_{\mu\nu} = \sigma_{,\mu\nu} - \tfrac{1}{2}\sigma_{,\mu}\sigma_{,\nu} + \tfrac{1}{2}\eta_{\mu\nu}\left(\eta^{\alpha\beta}\sigma_{,\alpha\beta} + \eta^{\alpha\beta}\sigma_{,\alpha}\sigma_{,\beta}\right) \tag{2}$$

Here a comma indicates ordinary differentiation. Einstein's field equations are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = -8\pi T_{\mu\nu} \tag{3}$$

where $T_{\mu\nu}$ stands for the energy-momentum tensor. For any trace-free energy-momentum tensor we have $R = 0$ from the field equations (2) and thus in view of (1) the most general form of σ subject to the restriction that $\sigma_{,\mu}$ is a null vector, can be obtained as the solution of the equation

$$\eta^{\alpha\beta}\sigma_{,\alpha\beta} = \Box\sigma = 0 \tag{4}$$

where $\Box\sigma$ represents the d'Alembertian in Minkowski coordinates.

In the next step we consider a particular form of the energy-momentum tensor where it is expressed as

$$8\pi T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} \tag{5}$$

where $\phi_{,\mu}$ is the gradient of ϕ. We get this form of energy-momentum tensor for a massless long-range scalar meson field, and conformally flat solutions corresponding to this scalar field were previously obtained by others (Penny, 1976; Som and Santos, 1978). We use the same form of energy-momentum tensor to describe a radiation field so that $\phi_{,\mu}$ is a null field and the special class of solutions may be obtained for $\phi = \phi(\sigma)$. In view of the fact that we choose $\phi_{,\mu}$ to be a null vector $\sigma_{,\mu}$ is also a null vector and so the function σ must satisfy the equation (4). It is possible to find the exact solution for Einstein–Maxwell field equations for the metric (1) and the energy-momentum tensor given in the form (5). In view of (2) the field equations (3) in such a case reduce to

$$R_{\mu\nu} = \sigma_{,\mu\nu} - \tfrac{1}{2}\sigma_{,\mu}\sigma_{,\nu} = -\phi'^2\sigma_{,\mu}\sigma_{,\nu} \tag{6}$$

where ϕ' represents $d\phi/d\sigma$. The equation (6) can now be written in the form

$$\sigma_{,\mu\nu} = \left[\tfrac{1}{2} - \phi'^2\right]\sigma_{,\mu}\sigma_{,\nu} \tag{7}$$

The first integration of (7) yields

$$\ln \sigma_{,\mu} = \xi(\sigma) + A \tag{8}$$

where $\xi(\sigma)$ is purely a function of σ and is given by

$$\xi(\sigma) = \int \left(\tfrac{1}{2} - \phi'^2\right) d\sigma$$

and A is an integration constant.

The second integration further enables one to write σ as an arbitrary function of ψ where ψ is a linear function of the coordinates. In other words

$$\sigma = \sigma(\psi)$$

where

$$\psi = a_\mu x^u \tag{9}$$

It is not difficult to see that the integration constants a_μ satisfy the condition

$$g^{\mu\nu} a_\mu a_\nu = 0 \tag{10}$$

a_μ's may be said to constitute a null vector with constant covariant components. A particular solution $e^\sigma = (a_\mu x^\mu + 1)$ was previously obtained by Penny in a completely different context, while discussing a scalar field.

3. MAXWELL FIELD TENSORS

The equations (6) and (9) allow us to write the field equations for the radiation field under consideration as

$$R_{\mu\nu} = -2q a_\mu a_\nu \tag{11}$$

where $2q = (\frac{1}{2}\sigma_\psi^2 - \sigma_{\psi\psi})$, and σ_ψ stands for $d\sigma/d_\psi$. q is therefore a function of ψ and may be said to be a measure of the energy density of the radiation field and a_μ is said to be the propagation vector of the radiation field

(Misner et al., 1965). It is easy to see that the null vector a_μ satisfies the relations

$$a^{\mu}{}_{;\mu} = a_{\mu;\nu}a^{\nu} = a_{(\mu;\nu)}a^{\mu;\nu} = a_{[\mu;\nu]}a^{\mu;\nu} = 0 \tag{12}$$

and so the vector a_μ may be said to represent geodetic, shear-free, divergenceless, and twist-free null congruence.

The most general form of $f^{\mu\nu}$ subject to the change of amplitude and polarization factors suitable for the energy-momentum tensor given in (11) may be written as

$$F^{\mu\nu} = (q)^{1/2}[a^{\mu}(m^{\nu}\cos\beta + n^{\nu}\sin\beta) - a^{\nu}(m^{\mu}\cos\beta + n^{\mu}\sin\beta)] \tag{13}$$

where m^μ and n^μ are unit spacelike vectors orthogonal to a^μ. m^μ and n^μ themselves are mutually orthogonal. $(q)^{1/2}$ and β are the amplitude and polarization factors. Here one can choose the spatial direction of the radiation flow along the x^3 axis, so that without loss of generality one may take $a_1 = a_2 = 0$ and $a_4 = -a_3 = 1$. Further one can choose the nonvanishing components m^μ and n^μ to be m' and n^2, respectively, so that $m^1 = n^2 = e^{-\sigma/2}$. These choices of m^μ and n^μ are possible without loss of generality, for one can start with m^μ and n^μ having components along both the x^1 and x^2 axes and finally arriving at the same electromagnetic field tensors given below for the electromagnetic field. The line element is then

$$ds^2 = e^{\sigma}\left[(dx^4)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2\right] \tag{14}$$

with e^σ as any arbitrary function of $(x^4 - x^3)$. With a coordinate transformation $u = (x^4 - x^3)$ the line element (14) can be otherwise written in a simple form

$$ds^2 = e^{\sigma(u)}\left[du^2 + 2du\,dx^3 - (dx^1)^2 - (dx^2)^2\right] \tag{15}$$

x^3 is now a null coordinate. The surviving components of the Maxwell field tensors are from (13)

$$\begin{aligned} F^{31} &= F^{41} = (q)^{1/2}e^{-3\sigma/2}\cos\beta \\ F^{32} &= F^{42} = -(q)^{1/2}e^{-3\sigma/2}\sin\beta \end{aligned} \tag{16}$$

In (16) q is a function of u. The field tensors satisfy Maxwell equations

$$F^{\mu\nu}{}_{;\nu} = F_{[\mu\nu,\alpha]} = 0$$

provided

$$\beta_{,1} = \beta_{,2} = \beta_{,3} = 0$$

and β can thus be an arbitrary function of only u. The metric (15) therefore represents a null electromagnetic field representing plane waves with e.m. field tensors given by (16). The arbitrariness in the polarization factor indicates that the field tensors for the null field are not unique (Witten, 1962). Further, it may also be mentioned that the propagation vector satisfies Robinson's conditions (Robinson, 1961) that it describes null congruence, which must be shear-free and geodetic.

ACKNOWLEDGMENTS

Our thanks are due to Professor A. Papapetrou for reading the manuscript and for some useful discussions. The authors acknowledge also the financial support of F.I.N.E.P of Brasil.

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